Solutions 2

Exercise 3.10

$$\mathbf{I}(\boldsymbol{\theta}) = \mathbb{E}\left[\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \cdot \frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})^T}{\partial \boldsymbol{\theta}}\right]$$

$$\mathbf{a}^T\mathbf{I}(\boldsymbol{\theta})\mathbf{a} = \mathbb{E}\left[\mathbf{a}^T\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \cdot \frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})^T}{\partial \boldsymbol{\theta}}\mathbf{a}\right]$$

$$= \mathbb{E}\left[(\mathbf{a}^T\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}})^2\right] \geq 0$$

for any $\mathbf{a}$ and all $\boldsymbol{\theta}$, thus it is positive semi-definite. For Problem 3.3, letting $\boldsymbol{\theta} = [\ A \quad r\ ]^T$ and using (3.31) in Page 47, we can get

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = \frac{1}{\sigma^2}[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_i}]^T[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_j}]$$

where $\boldsymbol{\mu}(\boldsymbol{\theta}) = [\ A \quad Ar \quad \ldots \quad Ar^{N-1}\ ]$. Thus,

$$\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial A} = [\ 1 \quad r \quad \ldots \quad r^{N-1}\ ]^T \qquad \frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial r} = A[\ 0 \quad 1 \quad \ldots \quad (N-1)r^{N-2}\ ]^T$$

$$\mathbf{I}(\boldsymbol{\theta}) = \frac{1}{\sigma^2}\begin{bmatrix} \sum_{n=0}^{N-1} r^{2n} & A\sum_{n=0}^{N-1} nr^{2n-1} \\ A\sum_{n=0}^{N-1} nr^{2n-1} & A^2\sum_{n=0}^{N-1} n^2r^{2n-2} \end{bmatrix}$$

If $A = 0$, $\mathbf{I}(\boldsymbol{\theta})$ is not positive definite.

Exercise 3.14

Since $x[n] \sim \mathcal{N}(A, \sigma_A^2)$, we can get $\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$.

$$\mathbb{E}(\hat{A}|A = A_0) = A_0 \qquad \text{var}(\hat{A}|A = A_0) = \frac{\sigma^2}{N} \to 0, N \to \infty$$

So $\hat{A} \to A_0$ when $N \to \infty$. Consider the estimator $\hat{\sigma}_A^2 = (\hat{A})^2$, we have $\mathbb{E}[\hat{\sigma}_A^2] = \mathbb{E}[\hat{A}^2] =$

$\text{var}(\hat{A}) + \mathbb{E}[\hat{A}]^2 = A_0^2 + \frac{\sigma^2}{N} \to A_0^2$ when $N \to \infty$. Since $\mathbb{E}[A^2] = \text{var}[A] = \sigma_A^2$ and

$$\begin{aligned}
\mathbb{E}[A^4] &= \int_{-\infty}^{\infty} A^4 \frac{1}{\sqrt{2\pi\sigma_A^2}} e^{-\frac{A^2}{2\sigma_A^2}} dA \\
&= \frac{1}{\sqrt{2\pi\sigma_A^2}} \int_{-\infty}^{\infty} A^3(-\sigma_A^2) de^{-\frac{A^2}{2\sigma_A^2}} \\
&= \frac{1}{\sqrt{2\pi\sigma_A^2}} \cdot \left[A^3(-\sigma_A^2) e^{-\frac{A^2}{2\sigma_A^2}} |_{-\infty}^{\infty} + 3 \int_{-\infty}^{\infty} A^2 \sigma_A^2 e^{-\frac{A^2}{2\sigma_A^2}} dA \right] \\
&= 3\sigma_A^2 \int_{-\infty}^{\infty} A^2 \frac{1}{\sqrt{2\pi\sigma_A^2}} e^{-\frac{A^2}{2\sigma_A^2}} dA \\
&= 3\sigma_A^4
\end{aligned}$$

we can get $\text{var}(\hat{\sigma}_A^2) = \text{var}(\hat{A}^2) = \mathbb{E}[A^4] - \mathbb{E}[A^2]^2 = 2\sigma_A^4$. So that $\text{var}(\hat{\sigma}_A^2) \to 2\sigma_A^4$, which is just the CRLB as $N \to \infty$. The $\hat{\sigma}_A^2$ cannot be estimated without error because we cannot reduce the random nature of A when we only have one realization of A.

Exercise 3.15

Since $x[n]$s are independent, we have $\mathbf{I}(\rho) = N\mathbf{i}(\rho)$, where $\mathbf{i}(\rho)$ is the Fisher information matrix for a single vector sample. Using (3.32), we can get

$$\mathbf{i}(\rho) = \frac{1}{2}\text{tr}\left[\left(\mathbf{C}^{-1}(\rho) \frac{\partial \mathbf{C}(\rho)}{\partial \rho} \right)^2 \right]$$

and

$$\mathbf{C}^{-1}(\rho) = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \qquad \frac{\partial \mathbf{C}(\rho)}{\partial \rho} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus,

$$\mathbf{i}(\rho) = \frac{1+\rho^2}{(1-\rho^2)^2} \qquad \mathbf{I}(\rho) = \frac{N(1+\rho^2)}{(1-\rho^2)^2}$$

We can get CRLB for ρ as $\text{var}(\rho) \geq \frac{(1-\rho^2)^2}{N(1+\rho^2)}$.

Exercise 3.16

Since we assume the mean of $x[n]$ is zero in Page 51, using (3.32), we can get

$$I(P_0) = \frac{1}{2}\text{tr}\left[\left(\mathbf{C}^{-1}(P_0) \frac{\partial \mathbf{C}(P_0)}{\partial P_0} \right)^2 \right]$$

Denote $r_{xx}[k] = \mathcal{F}^{-1}\{P_{xx}(f)\} = P_0\mathcal{F}^{-1}\{Q(f)\}$ and let $q[k] = \mathcal{F}^{-1}\{Q(f)\}$, the autocorrelation matrix of $\mathbf{x}$ has a Toeplitz form as follows:

$$[\mathbf{C}(P_0)]_{ij} = r_{xx}[i-j] = P_0 q[i-j] = P_0[\hat{\mathbf{C}}]_{ij} \qquad i, j = 1, 2, \ldots, N$$

where $[\hat{\mathbf{C}}]_{ij} = q[i-j]$. Hence,

$$\mathbf{C}^{-1}(P_0)\frac{\partial \mathbf{C}(P_0)}{\partial P_0} = \frac{1}{P_0}\hat{\mathbf{C}}^{-1}\hat{\mathbf{C}} = \frac{1}{P_0}\mathbf{E}$$

$$I(P_0) = \frac{1}{2}\mathrm{tr}[\frac{1}{P_0^2}\mathbf{E}^2] = \frac{N}{2P_0^2} \qquad \mathrm{var}(\hat{P}_0) \geq \frac{2P_0^2}{N}$$

Using (3.34) in Page 51, we can get the asymptotic form

$$\begin{aligned} I(P_0) &= \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial \ln P_{xx}(f;P_0)}{\partial P_0}\right)^2 df \\ &= \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial \ln P_0 Q(f)}{\partial P_0}\right)^2 df \\ &= \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{P_0^2} df \\ &= \frac{N}{2P_0^2} \end{aligned}$$

They are the same in this example, but not in general.

Exercise 3.17

Similar to example 3.14 in Page 56, we can get Fisher information matrix over the interval $n = -M, \ldots, 0, \ldots, M$ except $[\mathbf{I}(\boldsymbol{\theta})]_{23} = [\mathbf{I}(\boldsymbol{\theta})]_{32} = \frac{\pi A^2}{\sigma^2}\sum_{n=-M}^{n=M} n = 0$.

$$\mathbf{I}(\boldsymbol{\theta}) = \frac{1}{\sigma^2}\begin{bmatrix} \frac{2M+1}{2} & 0 & 0 \\ 0 & 2A^2\pi^2\sum_{n=-M}^{M} n^2 & 0 \\ 0 & 0 & \frac{(2M+1)A^2}{2} \end{bmatrix}$$

Thus, the CRLB is

$$\mathrm{var}(\hat{A}) \geq \frac{2\sigma^2}{2M+1} = \frac{2\sigma^2}{N} \qquad \mathrm{var}(\hat{\phi}) \geq \frac{2\sigma^2}{(2M+1)A^2} = \frac{1}{N\eta}$$

$$\mathrm{var}(\hat{f}_0) \geq \frac{\sigma^2}{2A^2\pi^2\sum_{n=-M}^{M} n^2} = \frac{6\sigma^2}{4A^2\pi^2 M(M+1)(2M+1)} = \frac{12}{(2\pi)^2\eta N(N^2-1)}$$

where $N = 2M+1$ and $\eta = A^2/(2\sigma^2)$. The results of $\hat{A}$ and $\hat{f}_0$ are the same as in example, however $\hat{\phi}$ is different.

Exercise 3.18

According to (3.40) in Page 56, we have

$$\overline{F^2} = \frac{\int_0^{T_s}(\frac{ds(t)}{dt})^2 dt}{\int_0^{T_s} s^2(t)dt} = \frac{\int_0^{0.02}(100)^2 dt}{\varepsilon} = \frac{200}{\varepsilon}$$

and

$$\mathrm{var}(\hat{R}) \geq \frac{c^2/4}{\frac{\varepsilon}{N_0/2}\overline{F^2}} = \frac{(1500)^2/4}{10^6 \cdot 200} = 0.00281$$